EE 477
Digital Signal Processing

6

FIR Frequency Response

FIR response to sinusoids

- The general definition of FIR:

$$y[n] = \sum_{k=0}^{M} b_k \cdot x[n-k] = \sum_{k=0}^{M} h[k] \cdot x[n-k]$$

- What if input is complex exponential?

$$x[n] = Ae^{j\phi} e^{j\hat{\omega}n}$$

$$y[n] = \sum_{k=0}^{M} b_k A e^{j\phi} e^{j\hat{\omega}(n-k)}$$

$$= Ae^{j\phi} e^{j\hat{\omega}n} \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

$$= Ae^{j\phi} e^{j\hat{\omega}n} H(\hat{\omega})$$

Frequency Response

- Note the result carefully:

$$y[n] = Ae^{j\phi}e^{j\hat{\omega}n}H(\hat{\omega})$$

- <u>IF</u> the input is a <u>complex exponential</u>, the output is a complex exponential *with the same frequency*, but in general a different amplitude and phase as determined by $H(\omega)$: the *frequency response.*

Frequency response (cont.)

- For FIR systems, the frequency response is determined by the coefficient sequence (which is just the impulse response sequence).

$$H(\hat{\omega}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

- The frequency response is a complex value for a particular frequency.

Frequency response (cont.)

- Polar formulation:

$$
\begin{aligned}
y[n] &= Ae^{j\phi} e^{j\hat{\omega}n} H(\hat{\omega}) \\
&= \left|H(\hat{\omega})\right| A \cdot e^{j(\angle H(\hat{\omega}) + \phi)} e^{j\hat{\omega}n}
\end{aligned}
$$

$$
\left|H(\hat{\omega})\right| = \sqrt{(real)^2 + (imag)^2}
$$

$$
\angle H(\hat{\omega}) = \arctan\left(\frac{imag}{real}\right)
$$

Frequency Response (cont.)

- Example:

$$y[n] = x[n] + 4 \cdot x[n-1] + 3 \cdot x[n-2]$$

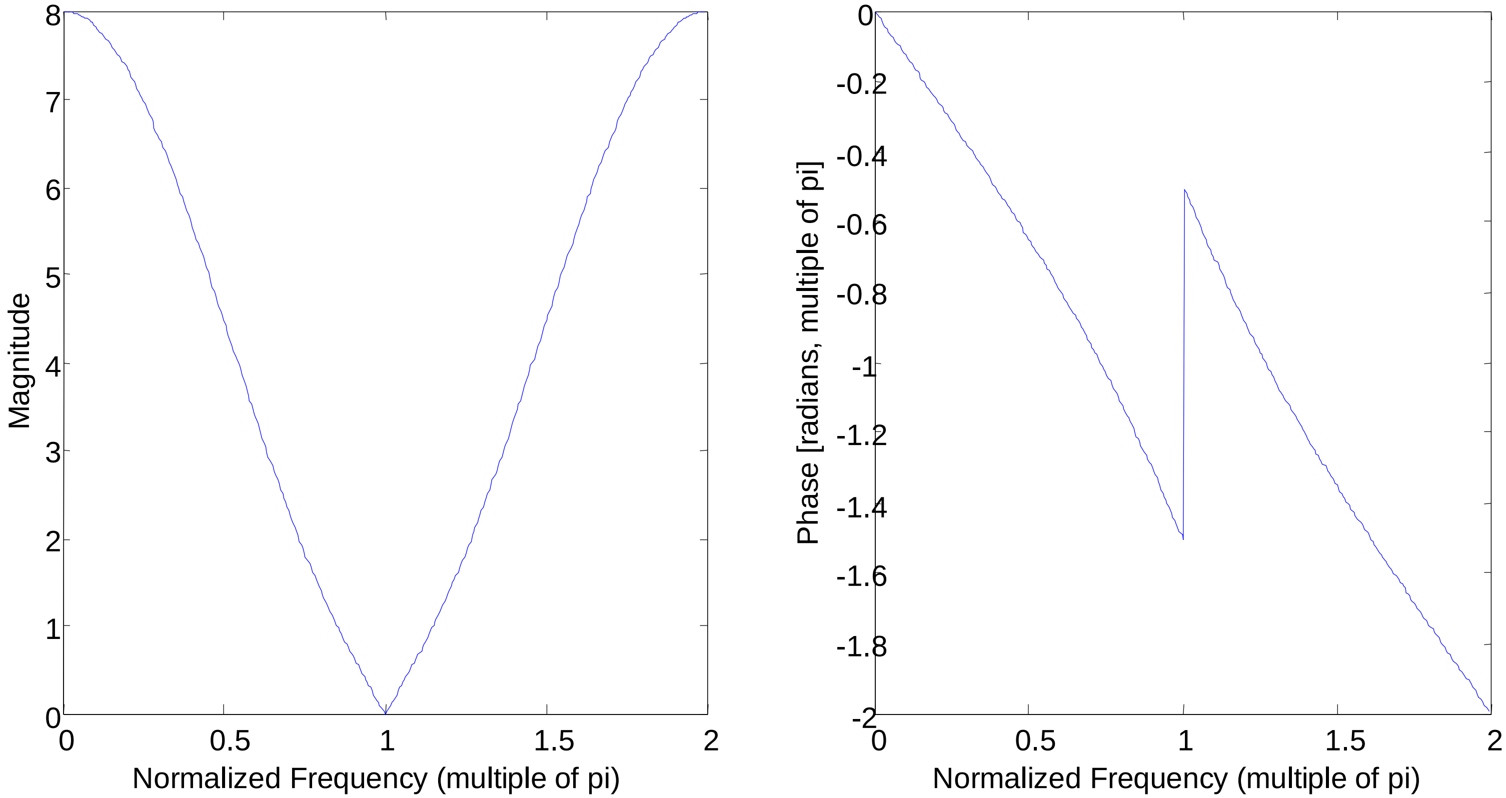
$$\{b_k\} = \{1, 4, 3\}$$

$$H(\hat{\omega}) = 1 + 4e^{-j\hat{\omega}} + 3e^{-j2\hat{\omega}}$$

$$|H(\hat{\omega})| = \left[(1 + 4\cos\hat{\omega} + 3\cos 2\hat{\omega})^2 + (4\sin\hat{\omega} + 3\sin 2\hat{\omega})^2\right]^{\frac{1}{2}}$$

$$\angle H(\hat{\omega}) = \arctan\left(\frac{-4\sin\hat{\omega} - 3\sin 2\hat{\omega}}{1 + 4\cos\hat{\omega} + 3\cos 2\hat{\omega}}\right)$$

Frequency Response (cont.)



Superposition

- If the input can be expressed as the sum of complex exponential signals, use the frequency response to determine the individual outputs, then add them up.
- This allows response determination in the *frequency domain*.

Transient and Steady State

- Note that our complex exponential is doubly infinite: all values of *n*
- Any practical system will need to *start* and then (probably) *stop* later
- Consider:

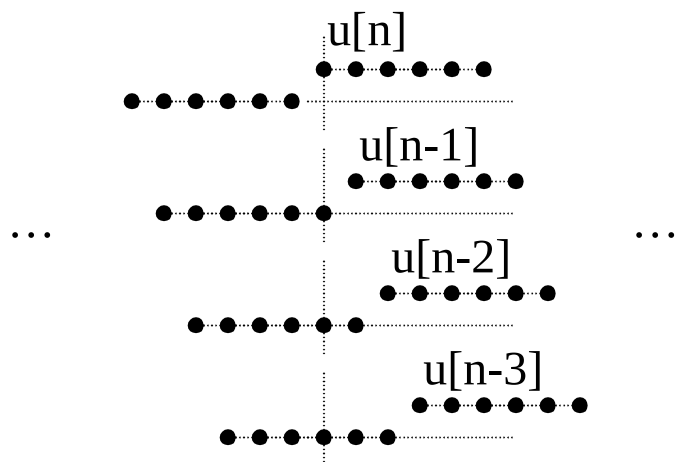
unit step: $u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

$$x[n] = Xe^{j\hat{\omega}n}u[n]$$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} b_k Xe^{j\hat{\omega}(n-k)}u[n-k]$$

Transient Response (cont.)

- Note the *M*+1 delayed u[n-k]:



Output is zero for n<0

$$y[n] = 0$$

Output is *transient* for $0 \le \text{n} < M$

$$y[n] = \left(\sum_{k=0}^{n} b_k e^{-j\hat{\omega}k} \right) X e^{j\hat{\omega}n}, \; 0 \le n < M$$

Output is same as ideal freq resp for $\text{n} \ge M$

$$y[n] = \left(\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} \right) X e^{j\hat{\omega}n}$$

Freq. Response Properties

- For FIR: $h[k] = b_k, 0 \le k \le M$

$$H(\hat{\omega}) = \sum_{k=0}^{M} h[k] e^{-j\hat{\omega}k}$$

- Note that $H(\omega)$ is always periodic in 2π
- If FIR coefs b_k are real, this implies that $H(\omega)$ has *conjugate symmetry:*

$$H(-\hat{\omega}) = H^*(\hat{\omega})$$

Conjugate Symmetry

- Conjugate symmetry $H(-\hat{\omega}) = H^*(\hat{\omega})$ indicates that the negative frequency portion of the spectrum is the complex conjugate of the positive frequency portion
- If we know one, we can calculate the other

Conjugate Symmetry Proof

$$H^*(\hat{\omega}) = \left(\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} \right)^*$$

$$= \sum_{k=0}^{M} b_k^* e^{+j\hat{\omega}k}$$

(b_k are real)

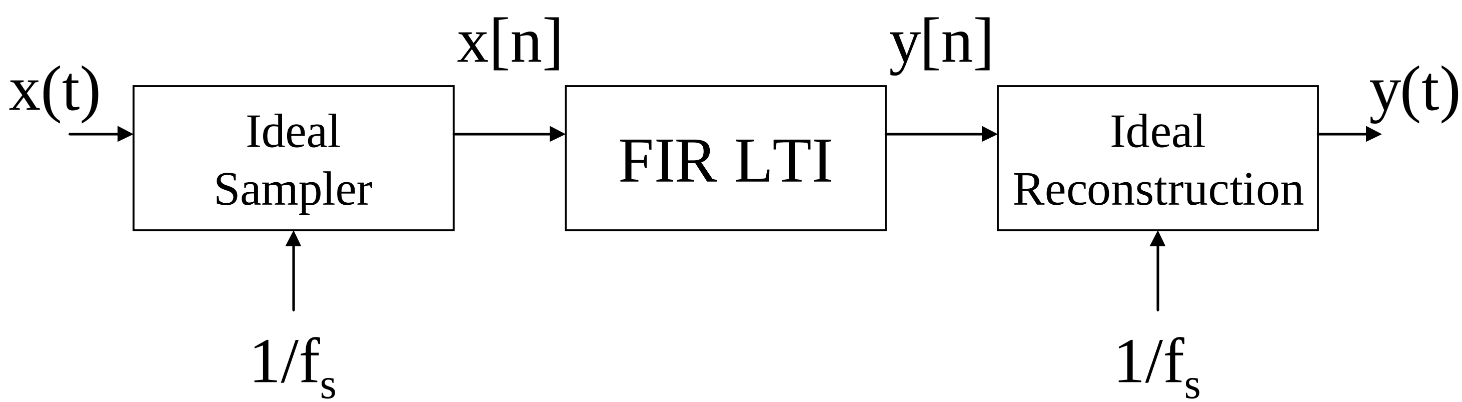
$$= \sum_{k=0}^{M} b_k e^{-j(-\hat{\omega})k} = H(-\hat{\omega})$$

Magnitude is an even function, Phase is odd

Real part is even, imaginary part is odd

Discrete time processing of continuous time signals

- Sample a continuous-time signal, perform discrete-time processing, then reconstruct the continuous-time signal



Discrete-time processing (cont.)

- Effect of sampling: assume

$$x(t) = Xe^{j\omega t}, \text{ sample at } t = nT_s$$

$$x[n] = Xe^{j\omega nT_s} = Xe^{j\hat{\omega}n}, \ \hat{\omega} = \omega T_s$$

$$y[n] = H(\hat{\omega})Xe^{j\hat{\omega}n} = H(\omega T_s)Xe^{j\omega nT_s}$$

$$y(t) = H(\omega T_s)Xe^{j\omega t}$$

- Overall response behaves like a continuous-time system with response $H(\omega T_s)$