EELE 477
Digital Signal Processing

7a

z-Transforms

z-Transform Definition

- For a finite length signal x[n], n=0…N, the z-transform is defined:

$$X(z) = \sum_{k=0}^{N} x[k]\, z^{-k}$$

$$= x[0]z^{0} + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + \ldots + x[N]z^{-N}$$

- *z* is simply a variable representing any *complex number*; that is, *z* is a *complex variable*
- X(z) is customarily expressed as a ratio of polynomials in z^{-k}

n-domain and *z*-domain

- Given x[n], we can use the formula to "take the z-transform" and get X(z).
- Given X(z), we can "take the inverse z-transform" and get x[n].
- Examples:

$$x[n] = \delta[n - n_0] \Leftrightarrow X(z) = z^{-n_0}$$

$$x[n] = 3\delta[n] - 6\delta[n-1] + 2\delta[n-2] + 7\delta[n-3]$$

$$\Leftrightarrow X(z) = 3 - 6z^{-1} + 2z^{-2} + 7z^{-3}$$

z-transforms and Linear Systems

- Consider FIR filter (convolve h and x):

$$y[n] = \sum_{k=0}^{M} h[k]\, x[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

- We previously applied $x[n]=e^{j\omega n}$, all n, to define the *frequency response.*
- Now apply $x[n]=z^n$:

$$y[n] = \sum_{k=0}^{M} b_k\, z^{(n-k)} = \sum_{k=0}^{M} b_k\, z^n z^{-k} = \underbrace{\left(\sum_{k=0}^{M} b_k\, z^{-k}\right)}_{H(z)} \underbrace{z^n}_{x[n]}$$

System Function

- Note the last result: $H(z) = \sum_{k=0}^{M} b_k \; z^{-k} = \sum_{k=0}^{M} h[k] \, z^{-k}$

  The system function H(z) is the z-transform of the unit sample response.
- System function (product) is related to convolution, since if x is z^n

$$y[n] = H(z) z^n$$

$$= h[n] * z^n$$

z-transform properties

- <u>Linearity</u>:
  $Z\{a x_1[n] + b x_2[n]\} = a Z\{x_1[n]\} + b Z\{x_2[n]\}$
- <u>Delay</u>: multiply by z^{-1} corresponds to delay of one sample; multiply by z^{-n0} corresponds to delay of n_0.
- <u>General infinite length z-transform</u>:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Note: need to verify that the sum converges!!

Z-transform and convolution

- Take the z-transform of a convolution expression:

$$y[n] = x[n] * h[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

$$Y(z) = Z\{h[0]x[n] + h[1]x[n-1] + \ldots + x[M]x[n-M]\}$$

$$= \sum_{k=0}^{M} h[k]\left(z^{-k}X(z)\right) = \left(\sum_{k=0}^{M} h[k]z^{-k}\right)X(z)$$

$$= H(z)X(z)$$

Sequence domain convolution $\Leftrightarrow$ *z-domain product*

z-transform: polynomial in z^{-1}

- Consider an FIR function:

$$y[n] = x[n] - 8.5x[n-1] + 13.5x[n-2] + 9x[n-3]$$

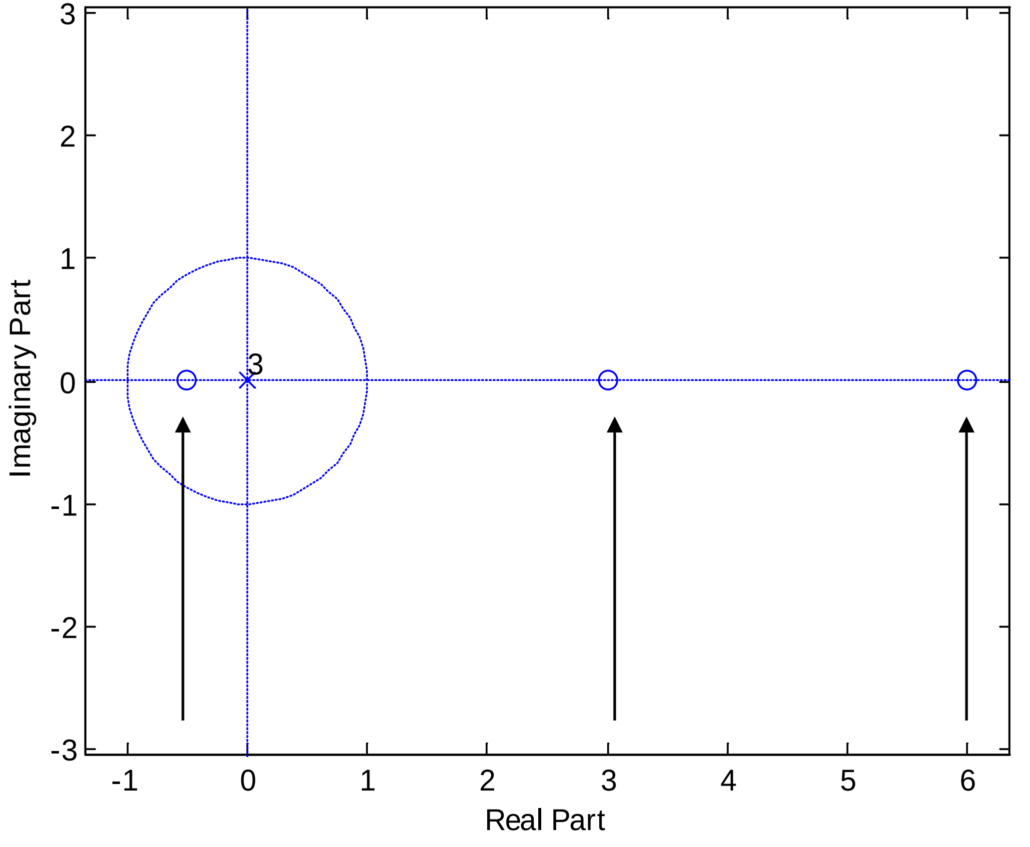
$$Y(z) = X(z) - 8.5z^{-1}X(z) + 13.5z^{-2}X(z) + 9z^{-3}X(z)$$

$$H(z) = Y(z)/X(z) = 1 - 8.5z^{-1} + 13.5z^{-2} + 9z^{-3}$$

$$= (1 + 0.5z^{-1})(1 - 3z^{-1})(1 - 6z^{-1})$$

- Zeros of H(z): -.5, 3, 6; 3 poles @ 0

Graphical Depiction of H(z)



Another h[n]↔H(z) example

- Consider an FIR system:

$$h[n] = \delta[n] - 2\delta[n-1] + 5\delta[n-2] + 5\delta[n-3] - 2\delta[n-4] + \delta[n-5]$$

$$H(z) = 1 - 2z^{-1} + 5z^{-2} + 5z^{-3} - 2z^{-4} + z^{-5}$$

$$= (1 - 2.618z^{-1} + 6.8541z^{-2})(1 - 0.382z^{-1} + 0.1459z^{-2})(1 + z^{-1})$$

- System zeros:

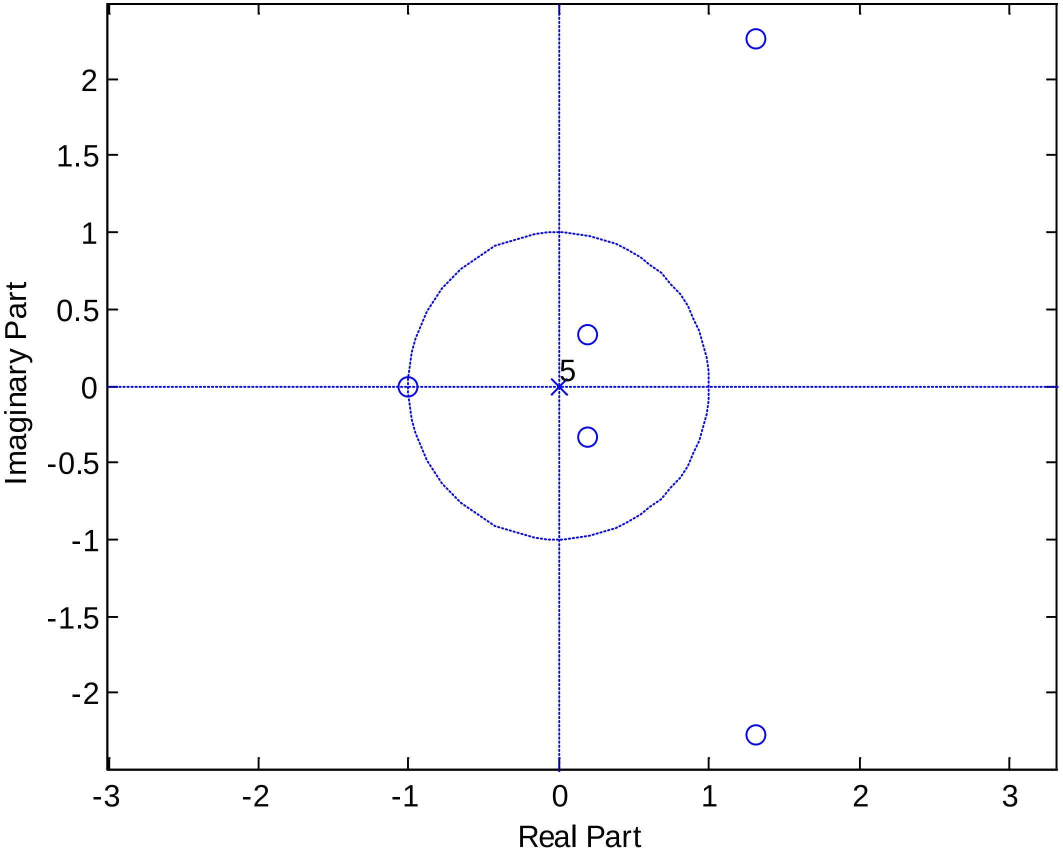
  1.3090 + j2.2673, 1.3090 - j2.2673

  -1.0000

  0.1910 + j0.3308, 0.1910 - j0.3308

Z-plane Sketch

Complex conjugate zeros⇒ real filter coefficients



Example z-transform product

- Ex: $x[n] = \delta[n-1] - 5\delta[n-2] + 8\delta[n-8] - 4\delta[n-9]$

  $h[n] = \delta[n] + \delta[n-1] - 0.3\delta[n-2]$

- Compute using z-transform product:

$$X(z) = z^{-1} - 5z^{-2} + 8z^{-8} - 4z^{-9}$$

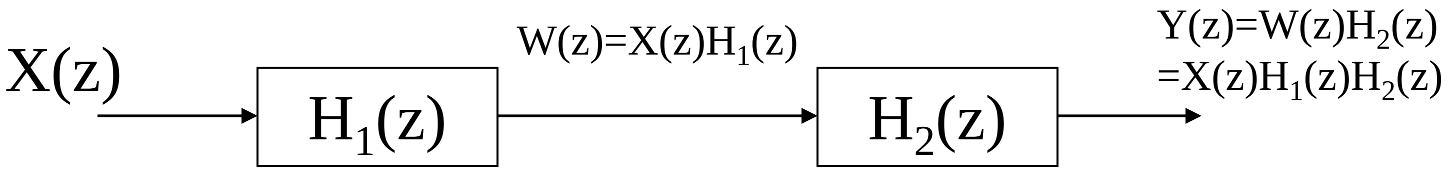
$$H(z) = 1 + z^{-1} - 0.3z^{-2}$$

$$Y(z) = X(z)H(z)$$

$$= z^{-1} - 4z^{-2} - 5.3z^{-3} + 1.5z^{-4} + 8z^{-8} + 4z^{-9} - 6.4z^{-10} + 1.2z^{-11}$$

Cascade of Systems

- Note cascade in terms of system functions:



- System function of cascade is the *product* of the system functions